

THE STUDY ON THE ANALYSIS OF THE EXISTENCE AND STABILITY OF SOLUTIONS TO THE NONLINEAR FRACTIONAL VOLTERRA-FREDHOLM INTEGRO- DIFFERENTIAL EQUATION

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ABSTRACT. This article investigates the existence, uniqueness, and stability of solutions for a class of nonlinear Volterra-Fredholm integro-differential equations with deviating arguments under initial conditions. By applying Banach and Schauder fixed-point theorems, sufficient conditions for the existence and uniqueness of solutions are established. Furthermore, Ulam-Hyers and Ulam-Hyers-Rassias stability criteria are derived. An example is provided to illustrate the applicability of the theoretical conclusions.

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1. INTRODUCTION

The exploration of fractional differential and integral equations is a relatively recent endeavor. Notable references in this area include the works of Kilbas et al. [1], Podlubny [2], and others. Integro-differential equations are of considerable interest due to their extensive applications across various scientific and technological fields, encompassing natural and engineering domains. These equations arise when the boundary values of a function are related to its values within the domain, carrying physical significance such as total mass and moments ([3, 5, 6, 7, 8]). Integral circumstances are more accurate gauges than local ones in some situations. See for a bibliography of recent works that detail the existence, uniqueness, and other characteristics of solutions to particular types of FIDEs. ([9, 10, 11, 12, 13, 14, 15]).

In their analysis of the initial value problem for Riemann-Liouville (R-L) nonlinear FDE and FIDE with deviating arguments, the authors considered the existence and uniqueness of solutions using the Banach fixed point theorem and the monotone iterative technique [23, 24].

Inspired by these problems, this paper delves into novel discussions concerning the existence, uniqueness, and stability outcomes for a nonlinear (R-L) fractional Volterra-Fredholm integro-differential equation. The equation incorporates deviating arguments

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and operates under an initial value problem, alongside a constant coefficient $\lambda > 0$

$$(1) \quad \mathfrak{D}_{0+}^{\mathfrak{w}} \mathbb{k}(\mathfrak{r}) = \lambda \mathbb{k}(\mathfrak{r}) + \Psi(\mathfrak{r}, \mathbb{k}(\mathfrak{r}), \mathbb{k}(\vartheta(\mathfrak{r})), \varphi \mathbb{k}(\mathfrak{r}), \varphi^* \mathbb{k}(\mathfrak{r})), \quad \mathfrak{r} \in \psi,$$

$$(2) \quad \mathbb{k}(0) = \mathbb{k}_0, \quad \mathbb{k}_0 > 0, \quad \psi = [0, \xi],$$

where $\mathfrak{D}^{\mathfrak{w}}(\cdot)$, $0 < \mathfrak{w} < 1$, is the (R-L) fractional derivative $0 < \mathfrak{w} < 1$, $\lambda > 0$, $\Psi \in \mathcal{C}([0, \xi] \times \mathbb{R}^4, \mathbb{R})$ are continuous functions $\vartheta \in \mathcal{C}(\psi \times \psi)$ and

$$\varphi \mathbb{k}(\mathfrak{r}) = \int_0^{\mathfrak{r}} \varphi_1(\mathfrak{r}, \sigma) \mathbb{k}(\sigma) d\sigma, \quad \varphi^* \mathbb{k}(\mathfrak{r}) = \int_0^{\xi} \varphi_2(\mathfrak{r}, \sigma) \mathbb{k}(\sigma) d\sigma.$$

The subsequent chapters of this paper are structured as follows: In Sect. 2, we introduce fundamental definitions and results. In Sect. 3, we utilize the Banach and Schauder fixed point theorems to establish the existence and uniqueness of solutions for problems (1)-(2). In Sect. 4, we employ the stability results for (R-L) fractional integro-differential equations with deviating arguments. In Sect. 5, we demonstrate our findings through relevant examples.

2. AUXILIARY RESULTS

Before presenting our primary results, we offer the essential definitions, preliminary details, and assumptions that will be employed in our subsequent discourse [4, 16, 17, 18, 19, 20, 21, 22].

Definition 2.1. ([1], [2]) For $\mathfrak{w} > 0$ is the integral

$$(3) \quad \mathfrak{I}_{0+}^{\mathfrak{w}} \mathbb{k}(\mathfrak{r}) = \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \mathbb{k}(\sigma) d\sigma,$$

is called the R-L fractional integral of order $\mathfrak{w}$.

Definition 2.2. ([1], [2]) The R-L fractional derivative $\mathfrak{w} > 0$, $(\nu - 1 < \mathfrak{w} < \nu)$ is defined as

$$(4) \quad \mathfrak{D}_{0+}^{\mathfrak{w}} \mathbb{k}(\mathfrak{r}) = \left(\frac{d}{d\mathfrak{r}} \right)^{\nu} (\mathfrak{I}_{0+}^{\nu-\mathfrak{w}} \mathbb{k}(\mathfrak{r})) = \frac{1}{\Gamma(\nu - \mathfrak{w})} \frac{d^{\nu}}{d\mathfrak{r}^{\nu}} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\nu-\mathfrak{r}-1} \mathbb{k}(\sigma) d\sigma.$$

Lemma 2.3. [1] Let $\mathfrak{r} \in \mathcal{C}^{\nu}[0, \xi]$, $\mathfrak{w} \in (\nu - 1, \nu)$, $\nu \in \mathbb{N}$. Then for all $\mathfrak{r} \in \psi$

$$\mathfrak{I}_{0+}^{\mathfrak{w}} \mathfrak{D}_{0+}^{\mathfrak{w}} \mathbb{k}(\mathfrak{r}) = \mathbb{k}(\mathfrak{r}) - \sum_{k=0}^{\nu-1} \frac{\mathfrak{r}^k}{k!} \mathbb{k}^{(k)}(0).$$

Theorem 2.4. [24] If a collection $\mathbb{k} = \{\mathbb{k}(\mathfrak{r}) \in \mathcal{C}(\psi, \mathbb{R})\}$ of functions satisfies uniform bounds and equicontinuity conditions on ψ , then there exists a subsequence $\{\mathbb{k}_{\nu}(\mathfrak{r})\}_{\nu=1}^{\infty}$ with in $\mathbb{k}$ that converges uniformly.

Lemma 2.5. [25] Let $\mathfrak{r} \in \mathcal{C}(\psi \times \mathbb{R})$ where for some $\mathfrak{r}_1 \in (0, \psi]$, $m(\mathfrak{r}_1) = 0$ & $m(\mathfrak{r}) \leq 0$, for $0 \leq \mathfrak{r} \leq \mathfrak{r}_1$. Then $D^{\mathfrak{r}} m(\mathfrak{r}_1) \geq 0$.

Lemma 2.6. [26] (Pachpatte's inequality) Let $\varpi(\mathfrak{r}), \varpi_0(\mathfrak{r}), \varpi_1(\mathfrak{r}) \in \mathcal{C}(\psi, \mathbb{R}_+)$ and let $\eta(\mathfrak{r}) \in \mathcal{C}(\psi, \mathbb{R}_+)$ be nondecreasing for which the inequality

$$\varpi(\mathfrak{r}) \leq \eta(\mathfrak{r}) + \int_0^{\mathfrak{r}} \varpi_0(\sigma) \varpi(\sigma) d\sigma + \int_0^{\mathfrak{r}} \varpi_0(\sigma) \int_0^{\sigma} \varpi_1(\tau) \varpi(\tau) d\tau d\sigma,$$

holds for any $\mathfrak{r} \in \mathfrak{R}_+$. Then

$$\varpi(\mathfrak{r}) \leq \eta(\mathfrak{r}) \left[1 + \int_0^{\mathfrak{r}} \varpi_0(\sigma) \exp \left(\int_0^{\sigma} (\varpi_0(\tau) + \varpi_1(\tau)) d\tau \right) d\sigma \right].$$

3. PRINCIPAL FINDINGS

This paragraph will present findings related to the existence and uniqueness of Equation (1) along with the initial value problem (2). Additionally, we will outline the assumptions that will guide our future analysis:

(A₁) There exist two constants φ^* and φ^ℓ such that

$$\varphi^* = \max\{|\varphi_1(\mathfrak{r}, \sigma)| : 0 \leq \sigma \leq \mathfrak{r} \leq \xi\}, \quad \varphi^\ell = \max\{|\varphi_2(\mathfrak{r}, \sigma)| : 0 \leq \sigma \leq \mathfrak{r} \leq \xi\}.$$

(A₂) There exist nonnegative constants δ , δ^* , δ_0 and δ_0^* such that function Ψ satisfies

$$\begin{aligned} |\Psi(\mathfrak{r}, \mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3, \mathbb{k}_4) - \Psi(\mathfrak{r}, \mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3, \mathfrak{H}_4)| &\leq \delta |\mathbb{k}_1 - \mathfrak{H}_1| + \delta^* |\mathbb{k}_2 - \mathfrak{H}_2| \\ &+ \delta_0 |\mathbb{k}_3 - \mathfrak{H}_3| + \delta_0^* |\mathbb{k}_4 - \mathfrak{H}_4|, \end{aligned}$$

for each $\mathfrak{r} \in \psi$, $\mathbb{k}_i, \mathfrak{H}_i \in \mathfrak{R}$, $i = 1, 2, 3, 4$.

(A₃) The function $\phi \in \mathcal{C}([0, \xi] \times \mathfrak{R}^4, \mathfrak{R})$, $\vartheta \in \mathcal{C}(\psi \times \psi)$, $\vartheta(\mathfrak{r}) \leq \mathfrak{r}$, $\mathfrak{r} \in \psi$.

(A₄) There exist $\mu \in \mathcal{C}([0, \xi], \mathfrak{R})$, and $\Xi : \mathfrak{R} \rightarrow \mathfrak{R}$ continuous function and non decreasing function such that

$$|\Psi(\mathfrak{r}, \kappa, \kappa^*, \kappa_0, \kappa_0^*)| \leq \mu(\mathfrak{r}) \Xi(|\kappa| + |\kappa^*| + |\kappa_0| + |\kappa_0^*|).$$

(A₅) Let $\|\mu^*\| = \max_{\mathfrak{r} \in \psi} |\mu(\mathfrak{r})|$ and there exists a constant $\omega > 0$ with

$$(5) \quad \frac{\omega}{[\mu^* \Xi(2\omega + \xi\omega(\varphi^\ell + \varphi^o)) + \omega\lambda] \mathcal{S} + \mathcal{S}^*} > 1,$$

where $\mathcal{S} = \frac{\xi^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)}$, $\mathcal{S}^* = \|\mathbb{k}_0\|$.

Lemma 3.1. Let $0 < \mathfrak{w} < 1$. Assume that $\mathbb{k} \in \mathcal{C}([0, \xi])$, $\mathbb{k}$ satisfies conditions (1)-(2) if and only if $\mathbb{k}$ satisfies the mixed type integral equation

$$\begin{aligned} \mathbb{k}(\mathfrak{r}) &= \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \left(\Psi(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma)), \int_0^{\sigma} \varphi_1(\sigma, \eta) \mathbb{k}(\eta) d\eta, \right. \\ (6) \quad &\left. \int_0^{\xi} \varphi_2(\sigma, \eta) \mathbb{k}(\eta) d\eta \right) d\sigma + \mathbb{k}_0 + \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \mathbb{k}(\sigma) d\sigma. \end{aligned}$$

Proof. Suppose $\mathbb{k}$ satisfies the problem (1), then from Lemma 2.3, we have

$$\mathfrak{I}_{0+}^{\mathfrak{w}} \mathfrak{D}_{0+}^{\mathfrak{w}} \mathbb{k}(\mathfrak{r}) = \mathbb{k}(\mathfrak{r}) - \sum_{k=0}^{\nu-1} \frac{\mathfrak{r}^k}{k!} \mathbb{k}^{(k)}(0).$$

As $\mathbb{k} \in (\psi, \mathfrak{R})$ so $\nu = 1$, on using $\nu = 1$ in above equation, we have

$$(7) \quad \mathfrak{I}_{0+}^{\mathfrak{w}} \mathfrak{D}_{0+}^{\mathfrak{w}} \mathbb{k}(\mathfrak{r}) = \mathbb{k}(\mathfrak{r}) - \mathbb{k}(0).$$

From first equation of problem (1), we have

$$\mathfrak{D}_{0+}^{\mathfrak{w}} \mathbb{k}(\mathfrak{r}) = \lambda \mathbb{k}(\mathfrak{r}) + \Psi(\mathfrak{r}, \mathbb{k}(\mathfrak{r}), \mathbb{k}(\vartheta(\mathfrak{r}))), \int_0^{\mathfrak{r}} \varphi_1(\mathfrak{r}, \sigma) \mathbb{k}(\sigma) d\sigma, \int_0^{\xi} \varphi_2(\mathfrak{r}, \sigma) \mathbb{k}(\sigma) d\sigma,$$

using it in (7), we get

$$(8) \quad \mathbb{k}(\mathfrak{r}) - \mathbb{k}(0) = \mathfrak{I}_{0+}^{\mathfrak{w}} \left[\lambda \mathbb{k}(\mathfrak{r}) + \Psi(\mathfrak{r}, \mathbb{k}(\mathfrak{r}), \mathbb{k}(\vartheta(\mathfrak{r}))), \int_0^{\mathfrak{r}} \varphi(\mathfrak{r}, \sigma) \mathbb{k}(\sigma) d\sigma, \int_0^{\xi} \varphi_2(\mathfrak{r}, \sigma) \mathbb{k}(\sigma) d\sigma \right].$$

Conversely, if $\mathbb{k}$ satisfies (6), then we obtain the first equation of the problem through the (R-L) derivative of order $\mathfrak{w}$ of both sides, which is (1). We obtain the initial condition (2) when we plug $\mathfrak{r} = 0$ into equation (6). □

Theorem 3.2. Assume that the conditions for (Λ_1) and (Λ_2) are met.

$$(9) \quad \mathfrak{S} = \left[\frac{(\lambda + \delta + \delta^*) \xi^{\mathfrak{w}}}{\Gamma(\mathfrak{w} + 1)} + \frac{\delta_0 \varphi' \xi^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w} + 2)} + \frac{\delta_0^* \varphi^o \xi^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w} + 1)} \right] < 1.$$

Then there is a unique solution to the problems (1)-(2).

Proof. Define an operator $\Omega : \mathcal{C}(\psi, \mathfrak{R}) \rightarrow \mathcal{C}(\psi, \mathfrak{R})$ as

$$(10) \quad (\Omega \mathbb{k}(\mathfrak{r})) = \mathbb{k}_0 + \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \mathbb{k}(\sigma) d\sigma + \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \\ \times \Psi(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma))), \int_0^{\sigma} \varphi_1(\sigma, \eta) \mathbb{k}(\eta) d\eta, \int_0^{\xi} \varphi_2(\sigma, \eta) \mathbb{k}(\eta) d\eta d\sigma.$$

Currently, we will demonstrate that it acts as a contraction operator. Let's take any $\mathbb{k}, \mathbb{k}_0$

$$\begin{aligned} |\Omega \mathbb{k} - \Omega \mathbb{k}_0| &= \max_{\mathfrak{r} \in \psi} \|\Omega \mathbb{k}(\mathfrak{r}) - \Omega \mathbb{k}_0(\mathfrak{r})\| \\ &\leq \max_{\mathfrak{r} \in \psi} \left[\frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \left(\|\Psi(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma))), \int_0^{\sigma} \varphi_1(\sigma, \eta) \mathbb{k}(\eta) d\eta, \int_0^{\xi} \varphi_2(\sigma, \eta) \mathbb{k}(\eta) d\eta\| \right. \right. \\ &\quad \left. \left. - \|\Psi(\sigma, \mathbb{k}_0(\sigma), \mathbb{k}_0(\vartheta(\sigma))), \int_0^{\sigma} \varphi_1(\sigma, \eta) \mathbb{k}_0(\eta) d\eta, \int_0^{\xi} \varphi_2(\sigma, \eta) \mathbb{k}_0(\eta) d\eta\| \right) d\sigma + \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \|\mathbb{k}(\sigma) - \mathbb{k}_0(\sigma)\| d\sigma \right] \\ &\leq \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \|\mathbb{k}(\sigma) - \mathbb{k}_0(\sigma)\| d\sigma + \max_{\mathfrak{r} \in \psi} \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \\ &\quad \times \left[\delta \|\mathbb{k}(\sigma) - \mathbb{k}_0(\sigma)\| + \delta^* \|\mathbb{k}(\vartheta(\sigma)) - \mathbb{k}_0(\vartheta(\sigma))\| \right. \\ &\quad \left. + \delta_0 \varphi' \int_0^{\sigma} \|\mathbb{k}(\eta) - \mathbb{k}_0(\eta)\| d\eta + \delta_0^* \varphi^o \int_0^{\xi} \|\mathbb{k}(\eta) - \mathbb{k}_0(\eta)\| d\eta \right] d\sigma \\ &\leq \left[\frac{(\lambda + \delta + \delta^*) \xi^{\mathfrak{w}}}{\Gamma(\mathfrak{w} + 1)} + \frac{\delta_0 \varphi' \xi^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w} + 2)} + \frac{\delta_0^* \varphi^o \xi^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w} + 1)} \right] \|\mathbb{k} - \mathbb{k}_0\|_{\mathcal{C}} = \mathfrak{S} \|\mathbb{k} - \mathbb{k}_0\|_{\mathcal{C}} \end{aligned}$$

where

$$\mathfrak{S} = \left[\frac{(\lambda + \delta + \delta^*)\xi^{\mathfrak{w}}}{\Gamma(\mathfrak{w} + 1)} + \frac{\delta_0 \varphi' \xi^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w} + 2)} + \frac{\delta_0^* \varphi'' \xi^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w} + 1)} \right]$$

Consequently, by (9), Ω is a contraction. As a consequence of Banach's contraction principle, we conclude that Ω admits a unique fixed point, which is the solution of the system (1)-(2). $\square$

Theorem 3.3. *Assumptions $(A_1) - (A_5)$ are true, then the problem (1) - (2) admits at least one solution.*

Proof. We finalize the proof through a series of steps:

Step 1 : Ω is a continuous mapping.

Consider a sequence $\{\mathbb{k}_m\}$, converging to $\{\mathbb{k}\}$ as $m \rightarrow \infty$ in $\mathcal{C}(\psi, \mathfrak{R})$. For each $\mathfrak{r} \in \psi$, we can observe that:

$$\begin{aligned} & |\Omega \mathbb{k}_m(\mathfrak{r}) - \Omega \mathbb{k}(\mathfrak{r})| \\ &= \left| \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \mathbb{k}_m(\sigma) d\sigma + \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \right. \\ & \quad \times \Psi\left(\sigma, \mathbb{k}_m(\sigma), \mathbb{k}_m(\vartheta(\sigma)), \int_0^{\sigma} \varphi_1(\sigma, \eta) \mathbb{k}_m(\eta) d\eta, \int_0^{\xi} \varphi_2(\sigma, \eta) \mathbb{k}_m(\eta) d\eta\right) d\sigma \\ & \quad - \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \mathbb{k}(\sigma) d\sigma - \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \\ & \quad \times \Psi\left(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma)), \int_0^{\sigma} \varphi_1(\sigma, \eta) \mathbb{k}(\eta) d\eta, \int_0^{\xi} \varphi_2(\sigma, \eta) \mathbb{k}(\eta) d\eta\right) d\sigma \left. \right| \\ & \leq \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} |\mathbb{k}_m(\sigma) - \mathbb{k}(\sigma)| d\sigma \\ & \quad + \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \left(\left| \Psi(\sigma, \mathbb{k}_m(\sigma), \mathbb{k}_m(\vartheta(\sigma)), \int_0^{\sigma} \varphi_1(\sigma, \eta) \mathbb{k}_m(\eta) d\eta, \right. \right. \\ & \quad \left. \int_0^{\xi} \varphi_2(\sigma, \eta) \mathbb{k}_m(\eta) d\eta - \Psi(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma)), \int_0^{\sigma} \varphi_1(\sigma, \eta) \mathbb{k}(\eta) d\eta, \right. \\ & \quad \left. \left. \int_0^{\xi} \varphi_2(\sigma, \eta) \mathbb{k}(\eta) d\eta) \right| d\sigma. \end{aligned}$$

Since Ψ, φ_1 and φ_2 are continuous functions, we have

$$\|\Omega \mathbb{k}_m(\mathfrak{r}) - \Omega \mathbb{k}(\mathfrak{r})\| \rightarrow 0 \text{ as } m \rightarrow \infty$$

This means that Ω is continuous.

Step 2 Ω maps bounded sets into bounded sets in $\mathcal{C}(\psi, \mathfrak{R})$.

Let $\beta > 0$ and consider $\varphi_{\beta} = \{\mathbb{k} \in \mathcal{C}(\psi, \mathfrak{R}) : |\mathbb{k}| \leq \beta\}$, which represents a bounded set in $\mathcal{C}(\psi, \mathfrak{R})$. Now, for any $\mathbb{k} \in \varphi_{\beta}$ and $\mathfrak{r} \in \psi$, we can utilize condition $(A_4) - (A_5)$ and

equation (10) to establish the following:

$$\begin{aligned}
& |\Omega \mathbb{k}(\mathfrak{r})| \\
&= \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \left| \Psi(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma)), \int_0^{\sigma} \varphi_1(\sigma, \eta) \mathbb{k}(\eta) d\eta, \right. \\
&\quad \left. \int_0^{\xi} \varphi_2(\sigma, \eta) \mathbb{k}(\eta) d\eta) \right| d\sigma + \|\mathbb{k}_0\| + \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\xi} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \|\mathbb{k}(\sigma)\| d\sigma \\
&\leq \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \mu(\sigma) \Xi \left(\|\mathbb{k}(\sigma)\| + \|\mathbb{k}(\vartheta(\sigma))\| \right) \\
&\quad + \int_0^{\sigma} |\varphi_1(\sigma, \eta)| \|\mathbb{k}(\eta)\| d\eta, \int_0^{\xi} |\varphi_2(\sigma, \eta)| \|\mathbb{k}(\eta)\| d\eta \Big) d\sigma + \|\mathbb{k}_0\| \\
&\quad + \frac{\beta \lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} d\sigma \\
&\leq \mu^* \Xi \left(2\|\mathbb{k}\| + \xi \Phi^{\iota} \|\mathbb{k}\| + \xi \Phi^o \|\mathbb{k}\| \right) \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} d\sigma + \|\mathbb{k}_0\| \\
&\quad + \frac{\beta \lambda \xi^{\mathfrak{w}}}{\mathfrak{w} \Gamma(\mathfrak{w})} \\
&\leq \|\mathbb{k}_0\| + \frac{\beta \lambda \xi^{\mathfrak{w}}}{\Gamma(\mathfrak{w} + 1)} + \frac{\|\mu^*\| \Xi \left(2\beta + \xi \beta (\varphi^{\iota} + \varphi^o) \right) \xi^{\mathfrak{w}}}{\Gamma(\mathfrak{w} + 1)} \\
&\leq \|\mathbb{k}_0\| + \frac{\left[\|\mu^*\| \Xi \left(2\beta + \xi \beta (\varphi^{\iota} + \varphi^o) \right) + \beta \lambda \right] \xi^{\mathfrak{w}}}{\Gamma(\mathfrak{w} + 1)}.
\end{aligned}$$

Thus, for any $\mathbb{k} \in \varphi_{\beta}$, then

$$\|\Omega \mathbb{k}\| \leq \mathcal{S}^* + [\|\mu^*\| \Xi (2\beta + \xi \beta (\varphi^{\iota} + \varphi^o)) + \beta \lambda] \mathcal{S}.$$

This condition holds, providing evidence for our statement.

Step 3 Ω maps bounded sets into equicontinuous sets of in $\mathcal{C}(\psi, \mathfrak{R})$.

Suppose $\mathbb{k}$ belongs to φ_{β} and $\mathfrak{r}_1, \mathfrak{r}_2$ are elements of ψ such that $\mathfrak{r}_1 < \mathfrak{r}_2$. By utilizing condition $(A_4) - (A_5)$ and equation (10) we obtain the following result

$$\begin{aligned}
& \|\Omega \mathbb{k}(\mathfrak{r}_1) - \Omega \mathbb{k}(\mathfrak{r}_2)\| \\
&\leq \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}_1} [(\mathfrak{r}_1 - \sigma)^{\mathfrak{w}-1} - (\mathfrak{r}_2 - \sigma)^{\mathfrak{w}-1}] \|\mathbb{k}(\sigma)\| d\sigma \\
&\quad + \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}_1} [(\mathfrak{r}_1 - \sigma)^{\mathfrak{w}-1} - (\mathfrak{r}_2 - \sigma)^{\mathfrak{w}-1}] \\
&\quad \times \left\| \Psi(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma)), \int_0^{\sigma} \varphi_1(\sigma, \eta) \mathbb{k}(\eta) d\eta, \int_0^{\xi} \varphi_2(\sigma, \eta) \mathbb{k}(\eta) d\eta) \right\| d\sigma \\
&\quad + \frac{\lambda}{\Gamma(\mathfrak{w})} \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} (\mathfrak{r}_2 - \sigma)^{\mathfrak{w}-1} \|\mathbb{k}(\sigma)\| d\sigma + \frac{1}{\Gamma(\mathfrak{w})} \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} (\mathfrak{r}_1 - \sigma)^{\mathfrak{w}-1} \\
&\quad \times \left\| \phi(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma)), \int_0^{\sigma} \varphi_1(\sigma, \eta) \mathbb{k}(\eta) d\eta, \int_0^{\xi} \varphi_2(\sigma, \eta) \mathbb{k}(\eta) d\eta) \right\| d\sigma
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\beta\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}_1} [(\mathfrak{r}_1 - \sigma)^{\mathfrak{w}-1} - (\mathfrak{r}_2 - \sigma)^{\mathfrak{w}-1}] d\sigma \\
&+ \frac{\mu^* \Xi \left(2\beta + \xi\beta(\varphi^\iota + \varphi^o) \right)}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}_1} [(\mathfrak{r}_1 - \sigma)^{\mathfrak{w}-1} - (\mathfrak{r}_2 - \sigma)^{\mathfrak{w}-1}] d\sigma \\
&+ \frac{\beta\lambda}{\Gamma(\mathfrak{w})} \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} [(\mathfrak{r}_1 - \sigma)^{\mathfrak{w}-1} d\sigma \\
&+ \frac{\mu^* \Xi \left(2\beta + \xi\beta(\varphi^\iota + \varphi^o) \right)}{\Gamma(\mathfrak{w})} \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} [(\mathfrak{r}_1 - \sigma)^{\mathfrak{w}-1} d\sigma \\
&\leq \frac{\lambda\beta + \sigma^* \Xi \left(2\beta + \xi\beta(\Phi^\iota + \Phi^o) \right)}{\Gamma(\mathfrak{w} + 1)} [2(\mathfrak{r}_2 - \mathfrak{r}_1)^{\mathfrak{w}} + (\mathfrak{r}_1^{\mathfrak{w}} - \mathfrak{r}_2^{\mathfrak{w}})] \\
&\longrightarrow 0 \text{ as } \mathfrak{r}_1 \rightarrow \mathfrak{r}_2.
\end{aligned}$$

The assertion that $\Omega : \mathcal{C}(\psi, \mathfrak{R}) \rightarrow \mathcal{C}(\psi, \mathfrak{R})$ is continuous and entirely continuous can be made using the Arzela-Ascoli Theorem 2.4.

Lastly $\Omega(\varphi_\beta) \subset \varphi_\beta$

Let $\mathbb{k} \in \varphi_\beta$ we demonstrate that $\Omega(\varphi_\beta) \in \varphi_\beta$ for any $\mathfrak{r} \in \psi$, utilizing condition (A_4) – (A_5) and equation (10) to establish the following.

$$\begin{aligned}
|\Omega \mathbb{k}(\mathfrak{r})| &= \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \left| \Psi(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma))), \int_0^\sigma \varphi_1(\sigma, \eta) \mathbb{k}(\eta) d\eta, \right. \\
&\quad \left. \int_0^\xi \varphi_2(\sigma, \eta) \mathbb{k}(\eta) d\eta \right| d\sigma + \|\mathbb{k}_0\| + \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^\xi (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} |\mathbb{k}(\sigma)| d\sigma \\
&\leq \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \mu(\sigma) \Xi \left(|\mathbb{k}(\sigma)| + |\mathbb{k}(\vartheta(\sigma))| + \int_0^\sigma |\varphi_1(\sigma, \eta)| |\mathbb{k}(\eta)| d\eta, \right. \\
&\quad \left. \int_0^\xi |\varphi_2(\sigma, \eta)| |\mathbb{k}(\eta)| d\eta \right) d\sigma + \|\mathbb{k}_0\| + \frac{\beta\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} d\sigma \\
&\leq \mu^* \Xi \left(2\|\mathbb{k}\| + \xi\varphi^\iota \|\mathbb{k}\| + \xi\varphi^o \|\mathbb{k}\| \right) \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} d\sigma + \|\mathbb{k}_0\| + \frac{\beta\lambda\xi^{\mathfrak{w}}}{\mathfrak{w}\Gamma(\mathfrak{w})} \\
&\leq \|\mathbb{k}_0\| + \frac{\omega\lambda\xi^{\mathfrak{w}}}{\Gamma(\mathfrak{w} + 1)} + \frac{\|\mu^*\| \Xi \left(2\omega + \xi\omega(\varphi^\iota + \varphi^o) \right) \xi^{\mathfrak{w}}}{\Gamma(\mathfrak{w} + 1)} \\
&\leq \|\mathbb{k}_0\| + \frac{\left[\|\mu^*\| \Xi \left(2\omega + \xi\omega(\varphi^\iota + \varphi^o) \right) + \beta\lambda \right] \xi^{\mathfrak{w}}}{\Gamma(\mathfrak{w} + 1)}.
\end{aligned}$$

Therefor

$$\|\Omega \mathbb{k}\| \leq \mathcal{S}^* + \mu^* \Xi \left[\|\mu^*\| \Xi (2\omega + \xi\omega(\varphi^\iota + \varphi^o)) + \beta\lambda \right] \mathcal{S}$$

by the inequality (5), we have

$$\|\Omega \mathbb{k}\| \leq \omega,$$

and hence

$$\Omega(\varphi_\beta) \subset \varphi_\beta$$

The Arzela-Ascoli theorem shows that Ω is relatively compact in both scenarios, and Schauder's fixed point theorem states that Ω has a fixed point. Then, Ω is a solution of the system (1)-(2). $\square$

4. STABILITY RESULTS

We investigate the problem (1)-(2) is Ulam stability in this section. The following issue is now being considered for Ulam stability.

$$(11) \quad \mathfrak{D}_{0+}^{\mathfrak{w}} \mathbb{k}(\mathfrak{r}) = \lambda \mathbb{k}(\mathfrak{r}) - \phi(\mathfrak{r}, \mathbb{k}(\mathfrak{r}), \mathbb{k}(\vartheta(\mathfrak{r})), \int_0^{\mathfrak{r}} \varphi_1(\mathfrak{r}, \sigma) \mathbb{k}(\sigma) d\sigma, \int_0^{\xi} \varphi_2(\mathfrak{r}, \sigma) \mathbb{k}(\sigma) d\sigma),$$

and the following inequalities

$$(12) \quad \left| \mathfrak{D}_{0+}^{\mathfrak{w}} \mathcal{F}(\mathfrak{r}) - \lambda \mathcal{F}(\mathfrak{r}) - \phi(\mathfrak{r}, \mathcal{F}(\mathfrak{r}), \mathbb{k}(\vartheta(\mathfrak{r})), \int_0^{\mathfrak{r}} \varphi_1(\mathfrak{r}, \sigma) \mathcal{F}(\sigma) d\sigma, \int_0^{\xi} \varphi_2(\mathfrak{r}, \sigma) \mathcal{F}(\sigma) d\sigma) \right| \leq \epsilon,$$

$$(13) \quad \left| \mathfrak{D}_{0+}^{\mathfrak{w}} \mathcal{F}(\mathfrak{r}) - \lambda \mathcal{F}(\mathfrak{r}) - \Psi(\mathfrak{r}, \mathcal{F}(\mathfrak{r}), \mathcal{F}(\vartheta(\mathfrak{r})), \int_0^{\mathfrak{r}} \varphi_1(\mathfrak{r}, \sigma) \mathcal{F}(\sigma) d\sigma, \int_0^{\xi} \varphi_2(\mathfrak{r}, \sigma) \mathcal{F}(\sigma) d\sigma) \right| \leq \epsilon \vartheta(\mathfrak{r}), \quad \mathfrak{r} \in \psi,$$

$$(14) \quad \left| \mathfrak{D}_{0+}^{\mathfrak{w}} \mathcal{F}(\mathfrak{r}) - \lambda \mathcal{F}(\mathfrak{r}) - \phi(\mathfrak{r}, \mathcal{F}(\mathfrak{r}), \mathcal{F}(\vartheta(\mathfrak{r})), \int_0^{\mathfrak{r}} \Phi_1(\mathfrak{r}, \sigma) \mathcal{F}(\sigma) d\sigma, \int_0^{\xi} \Phi_2(\mathfrak{r}, \sigma) \mathcal{F}(\sigma) d\sigma) \right| \leq \vartheta(\mathfrak{r}), \quad \mathfrak{r} \in \psi,$$

Definition 4.1. [19]. If a real number exists, the Eq. (11) is Ulam-Hyers stable. $\mathcal{C}_{\phi} > 0$ such that for each $\epsilon > 0$ and for each solution $\mathcal{F} \in \mathcal{C}(\psi, \mathfrak{R})$ of inequality (11) there exists a solution $\mathbb{k} \in \mathcal{C}(\psi, \mathfrak{R})$ of equation (11) with

$$|\mathcal{F}(\mathfrak{r}) - \mathbb{k}(\mathfrak{r})| \leq \mathcal{C}_{\phi}, \quad \mathfrak{r} \in \psi.$$

Definition 4.2. [19]. The equation given by (11) is of a general nature. If a solution exists that is Ulam-Hyers stable, denoted as $\psi_{\phi} \in ([0, \infty), 0, \infty)$ with $\psi_{\phi}(0) = 0$, then for each solution $\mathbb{k} \in \mathcal{C}(\psi, \mathfrak{R})$ of the inequality (12), a corresponding solution $\mathbb{k} \in \mathcal{C}(\psi, \mathfrak{R})$ of the equation (11) exists.

$$|\mathcal{F}(\mathfrak{r}) - \mathbb{k}(\mathfrak{r})| \leq \psi_{\phi}, \quad \mathfrak{r} \in \psi.$$

Definition 4.3. [19]. The equation denoted as (11) demonstrates Ulam-Hyers-Rassias stability concerning $\mathcal{C}_{\phi} \in \mathcal{C}(\psi, \mathfrak{R})$. This stability holds true if a real value $\mathcal{C}_{\phi} > 0$ exists, satisfying the condition that for each $\epsilon > 0$ and for every solution $\mathbb{k} \in \mathcal{C}(\psi, \mathfrak{R})$ of the inequality (13), there is a corresponding solution $\mathcal{F} \in \mathcal{C}(\psi, \mathfrak{R})$ of the equation (11).

$$|\mathbb{k}(\mathfrak{r}) - \mathcal{F}(\mathfrak{r})| \leq \epsilon \mathcal{C}_{\phi \vartheta(\mathfrak{r})}, \quad \mathfrak{r} \in \psi.$$

Theorem 4.4. *Supposing that conditions (A_1) , (A_2) , and (11) hold, the stability of the problem (1)-(2) under Ulam-Hyers criterion is established.*

Proof. For each $\epsilon > 0$, consider a function $\mathbb{k} \in \mathcal{C}(\psi, \mathfrak{R})$ that fulfills inequality (11); furthermore, consider the unique solution of the following issue as $\mathbb{k} \in \mathcal{C}(\psi, \mathfrak{R})$.

$$\mathfrak{D}_{0+}^{\mathfrak{w}} \mathbb{k}(\mathfrak{r}) = \lambda \mathbb{k}(\mathfrak{r}) + \Psi(\mathfrak{r}, \mathbb{k}(\mathfrak{r}), \mathbb{k}(\vartheta(\mathfrak{r})), \int_0^{\mathfrak{r}} \varphi_1(\mathfrak{r}, \sigma) \mathbb{k}(\sigma) d\sigma, \int_0^{\xi} \varphi_2(\mathfrak{r}, \sigma) \mathbb{k}(\sigma) d\sigma),$$

$$\mathbb{k}(0) = \mathbb{k}_0, \quad \mathfrak{r} \in \psi, \quad 0 < \mathfrak{w} \leq 1.$$

Using Lemma 3.1, we obtain

$$\begin{aligned} \mathbb{k}(\mathfrak{r}) &= \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \left(\Psi(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma)), \int_0^{\sigma} \varphi_1(\sigma, \eta) \mathbb{k}(\eta) d\eta, \right. \\ (15) \quad &\left. \int_0^{\xi} \varphi_2(\sigma, \eta) \mathbb{k}(\eta) d\eta) d\sigma + \mathbb{k}_0. \right. \end{aligned}$$

By integrating (11), we obtain:

$$\begin{aligned} &\left\| \mathbb{k}(\mathfrak{r}) - \mathbb{k}_0 - \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \mathbb{k}(\sigma) d\sigma - \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \right. \\ &\quad \left. \Psi\left(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma)), \int_0^{\sigma} \varphi_1(\mathfrak{r}, \eta) \mathbb{k}(\eta) d\eta, \int_0^{\xi} \varphi_2(\mathfrak{r}, \eta) \mathbb{k}(\eta) d\eta\right) d\sigma \right\| \\ (16) \quad &\leq \frac{\epsilon \mathfrak{r}^{\mathfrak{w}}}{\Gamma(\mathfrak{w} + 1)}. \end{aligned}$$

Using (A_1) , (A_2) and the inequality (16), for every $\mathfrak{r} \in \psi$, we have:

$$\begin{aligned} &\|\mathbb{k}(\mathfrak{r}) - \mathbb{k}(\mathfrak{r})\| \\ &\leq \left\| \mathbb{k}(\mathfrak{r}) - \mathbb{k}_0 - \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \mathbb{k}(\sigma) d\sigma - \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \right. \\ &\quad \left. \Psi\left(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma)), \int_0^{\sigma} \varphi_1(\eta, \eta) \mathbb{k}(\eta) d\eta, \int_0^{\xi} \varphi_2(\eta, \eta) \mathbb{k}(\eta) d\eta\right) d\sigma \right\| \\ &\quad + \left\| \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \|\mathbb{k}(\sigma) - \mathbb{k}(\sigma)\| d\sigma + \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \right. \\ &\quad \left. \Psi\left(\sigma, \mathbb{k}(\sigma), \mathbb{k}(\vartheta(\sigma)), \int_0^{\sigma} \varphi_1(\eta, \eta) \mathbb{k}(\eta) d\eta, \int_0^{\xi} \varphi_2(\eta, \eta) \mathbb{k}(\eta) d\eta\right) d\sigma \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\epsilon \mathfrak{r}^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)} + \frac{\lambda}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1} \|\mathbb{k}(\sigma) - \mathbb{k}(\sigma)\| d\sigma \\
&+ \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1} \left[\delta \|\mathbb{k}(\sigma) - \mathbb{k}(\sigma)\| + \delta^* \|\mathbb{k}(\vartheta(\sigma)) - \mathbb{k}(\vartheta(\sigma))\| \right. \\
&\left. + \delta_0 \int_0^{\sigma} \varphi_1(\sigma, \eta) \|\mathbb{k}(\eta) - \mathbb{k}(\eta)\| d\eta + \delta_0^* \int_0^{\xi} \varphi_2(\sigma, \eta) \|\mathbb{k}(\eta) - \mathbb{k}(\eta)\| d\eta \right] d\sigma.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\mathbb{k}(\mathfrak{r}) - \mathbb{k}(\mathfrak{r})\| &\leq \frac{\epsilon \mathfrak{r}^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)} + \frac{(\lambda + \delta + \delta^*)}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1} \|\mathbb{k}(\sigma) - \mathbb{k}(\sigma)\| d\sigma \\
&+ \frac{(\varphi^{\iota} \delta_0 + \varphi^o \delta_0^*)}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1} \left(\int_0^{\sigma} \|\mathbb{k}(\eta) - \mathbb{k}(\eta)\| d\eta \right. \\
&\left. + \int_0^{\xi} \|\mathbb{k}(\eta) - \mathbb{k}(\eta)\| d\eta \right) d\sigma \\
&\leq \frac{\epsilon \mathfrak{r}^{\alpha}}{\Gamma(\mathfrak{w}+1)} + \int_0^{\mathfrak{r}} \frac{(\lambda + \delta + \delta^*)}{\Gamma(\mathfrak{w})} (\xi - \sigma)^{\mathfrak{w}-1} \left[\|\mathbb{k}(\sigma) - \mathbb{k}(\sigma)\| \right. \\
(17) \quad &\left. + \frac{(\varphi^{\iota} \delta_0 + \varphi^o \delta_0^*)}{(\lambda + \delta + \delta^*)} \int_0^{\sigma} \|\mathbb{k}(\eta) - \mathbb{k}(\eta)\| d\eta \right] d\sigma.
\end{aligned}$$

Utilizing the inequality provided in Theorem 3.3, attributed to Pachpatte, on equation (17), we derive the following result:

$$\begin{aligned}
&\|\mathbb{k}(\mathfrak{r}) - \mathbb{k}(\mathfrak{r})\| \\
&\leq \frac{\epsilon \mathfrak{r}^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)} \left[1 + \int_0^{\xi} \frac{(\lambda + \delta + \delta^*)}{\Gamma(\mathfrak{w})} (\xi - \sigma)^{\mathfrak{w}-1} \right. \\
&\times \exp \left(\int_0^{\sigma} \frac{(\lambda + \delta + \delta^*)}{\Gamma(\mathfrak{w})} (\xi - \sigma)^{\mathfrak{w}-1} + \frac{(\varphi^{\iota} \xi_1 + \varphi^o \delta_0^*)}{(\lambda + \delta + \delta^*)} d\eta \right) d\sigma \Big] \\
&\leq \frac{\epsilon \xi^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)} \left[1 + \int_0^{\xi} \frac{(\lambda + \delta + \delta^*)}{\Gamma(\mathfrak{w})} (\xi - \sigma)^{\mathfrak{w}-1} \right. \\
&\times \exp \left(\int_0^{\sigma} \left[\frac{(\lambda + \delta + \delta^*)}{\Gamma(\mathfrak{w})} (\xi - \eta)^{\mathfrak{w}-1} + \frac{(\varphi^{\iota} \delta_0 + \varphi^o \delta_0^*)}{(\lambda + \delta + \delta^*)} \right] d\eta \right) d\sigma \Big] \\
&= \epsilon C_{\phi}.
\end{aligned}$$

Thus the problem (1)-(2) is Ulam-Hyers stable. $\square$

Theorem 4.5. *Given that the conditions (Λ_1) , (Λ_2) , and (11) are satisfied, let's further assume the existence of an increasing function $\vartheta \in \mathcal{C}(\psi, \mathfrak{R}^+)$ and a positive constant $\Psi_{\vartheta} > 0$, such that the inequality $I^{\alpha} \vartheta(\mathfrak{r}) \leq \Psi_{\vartheta} \vartheta(\mathfrak{r})$ holds for any $t \in J$. With these assumptions in place, it follows that the nonlocal problem (1)-(2) exhibits Ulam-Hyers-Rassias stability.*

Proof. Equations (1)-(2) and the inequality (14) are investigated, building upon the conditions stated in Theorem 4.4. The Ulam-Hyers-Rassias stability of the issue (1)-(2) can be verified by conducting the same approach again. $\square$

5. APPLICATION

Consider the following problem:

$$(18) \quad \begin{aligned} \mathfrak{D}_{0+}^{0.5} \mathbb{k}(\mathfrak{r}) &= \frac{1}{5} |\mathbb{k}(\mathfrak{r})| + \mathfrak{r} + \frac{\mathfrak{r}^2 + 1}{20} \mathbb{k}(\mathfrak{r}) + \frac{\mathfrak{r}^2 + 1}{10} \mathbb{k}(\mathfrak{r}^{\frac{1}{2}}) \\ &+ \frac{1}{5} \int_0^{\mathfrak{r}} \frac{\cos(\sigma \mathfrak{r})}{\mathfrak{r}^2 + 3} |\mathbb{k}(\sigma)| d\sigma + \int_0^1 \frac{e^{-2\sigma \mathfrak{r}}}{(5+t)^2} |\mathbb{k}(\sigma)| d\sigma \end{aligned}$$

$$(19) \quad \mathbb{k}(0) = 1, \quad \psi = [0, 1].$$

Notice that the problem (18)-(19) can be treated as a special case of Eqs. (1)-(2) if we put $\mathfrak{w} = \frac{1}{2}$, $\xi = 1$, $\lambda = \frac{2}{3}$ and there are exist constants

$$\delta = \frac{1}{20}, \quad \delta^* = \frac{1}{10}, \quad \delta_0 = \frac{1}{15}, \quad \delta_0^* = \frac{1}{25}.$$

From equations(18)-(19) and inequality (9), we have

$$\mathfrak{S} = \left[\frac{(\lambda + \delta + \delta^*)\xi^{\mathfrak{w}}}{\Gamma(\mathfrak{w} + 1)} + \frac{\delta_0 \varphi' \xi^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w} + 2)} + \frac{\delta_0^* \varphi'' \xi^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w} + 1)} \right] \cong 0.49 < 1.$$

As a result, the equations (18)-(19) have a unique solution on ψ and are Ulam-Hyers stable according to Theorem 4.4.

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